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On the Completeness Problem of the Equations for Two-Layer Sedimentation Models

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Abstract: Based on the Saint-Venant-Exner equations for two-layer models the completeness problem and closing problem are discussed. It is shown that for the closing of the equations, in addition to the usual constitutive relations, it is required to explicitly specify one of the boundaries of the layers. For the simplest model, in the case where the free surface has a constant slope, we demonstrate the occurrence of the shock waves at the boundary between the layers.

Zusammenfassung: Basierend auf den Saint-Venant-Exner-Gleichungen für das Zwei-Schicht-Modell wird das Problem der Vollständigkeit und der Schließung der Gleichungen diskutiert. Es wird gezeigt, dass man eine der Grenzen der Schicht für die Schließung der Gleichungen unabhängig von den üblichen bestimmenden Beziehungen explizit vorgeben muss. Für das einfachste Modell wurde das Beispiel betrachtet, das die Entstehung der Stoßwellen an der Grenze zwischen den Schichten in dem Fall demonstriert, wenn die freie Oberfläche eine konstante Neigung hat.

INTRODUCTION

To describe the sedimentation process, the system of the Saint-Venant-Exner equations for the two-layer model and its various modifications are often used, see, for example, (PARKER 1982, PARKER et al. 1986, RIJN 1993, FALCINI et al. 2009, NADOLIN 2009, AUDUSSE et al. 2010, GAREGNANI 2011, FERNANDEZ-NIETO et al. 2014, 2015, MALDONADO & BORTHWICK 2016, ZHUKOV & SHIRYAEVA 2016, NADOLIN & ZHILYAEV 2017). It is assumed that the top layer is filled by the liquid containing suspended impurity (suspension) and the bottom layer is filled by liquid and sediment. To construct the equations the average over the thickness of the top layer is used. Equations for the bottom layer are based on phenomenological relations. Directly, the depth-averaged method does not allow us to construct closed mathematical model for the sedimentation process. To close the equations a sufficiently large number of constitutive relations is required for describing of the suspension velocity, the sediment velocity etc. Such relations, as rule, based on empirical relations, allow us to describe the sedimentation process in detail. In fact, the basic constitutive relations can be chosen on the basis of natural boundary conditions and the boundary conditions between the layers, i.e. without the empirical relations. It is enough to follow the general scheme: original equation + kinematic approximation + averaging + conditions at the boundaries of layers. However, even after determination of all the constitutive relations the equation system is not closed, since additional assumptions for one of the layer boundaries are required. Note that similar models were studied in previous works (PARKER 1982, PARKER et al. 1986). In particular, the model we present is similar to what proposed by PAOLA & VOLLER (2005), FERNANDEZ-NIETO et al. (2014, 2015).

In this paper the choices of one of boundaries as known functions are discussed. It is shown that the condition on the border between the layers plays an important role and in a basic example, when the free liquid surface has a constant slope, on the boundary between the layers can occur shock waves.

One of the aims of this paper is a demonstration of the fact that even the simplest models, based on basic principles, are good enough to qualitatively describe such complex processes as the occurrence of shock waves. We should say that the incompleteness of our simple model plays a dual role: on one hand, it shows some shortcomings of the model, more precisely, the failure of physical assumptions; on the other hand, it opens up opportunities to construct models at different levels of complexity.

BASIC EQUATIONS

To describe the sedimentation process in the fluid flow the two-layer model is used. We assume that the region is a composition of two layers $\mathbf{L}^t$ and $\mathbf{L}^b$ (Fig. 1). The top layer $\mathbf{L}^t$ is filled by the single-phase continuous medium (homogeneous) which consists of fluid and suspended impurities (suspension). The bottom layer $\mathbf{L}^b$ is filled by the multiphase (heterogeneous) continuous medium which consists of fluid and sediment. The layer boundaries are defined by the functions η^t, η^m, η^b (we note that in PARKER et al. (1986) the forms of the boundaries are determined by the grain size)

$$(1)\quad \mathbf{L}^t = \{(\mathbf{x},z) : \eta^m(\mathbf{x},t) \le z \le \eta^t(\mathbf{x},t)\},$$
$$\mathbf{L}^b = \{(\mathbf{x},z) : \eta^b(\mathbf{x},t) \le z \le \eta^m(\mathbf{x},t)\},$$
$$h^t = \eta^t - \eta^m,\ h^b = \eta^m - \eta^b,\ \mathbf{x} = (x,y).$$

Here, $h^t(\mathbf{x},t)$, $h^b(\mathbf{x},t)$ are the layer thickness $\mathbf{x} = (x,y)$ are the horizontal coordinates, z is the vertical coordinate, t is time.

Keywords: constitutive relations, averaging, kinematic approximation, shock waves.

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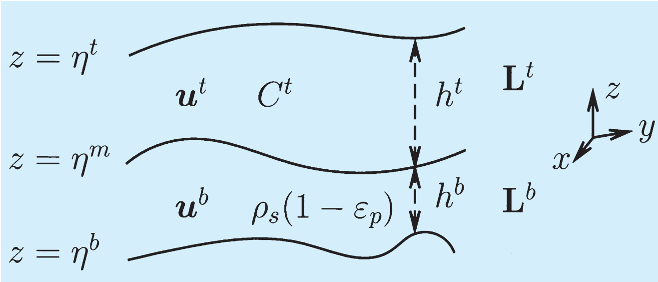

Fig. 1: The layers of the continuous medium. The top layer $\mathbf{L}^t$ is filled by fluid. The bottom layer $\mathbf{L}^b$ contains sediments (and fluid). The layer boundaries are defined by the functions η^t, η^m, η^b.

Abb. 1: Schichten der ununterbrochenen Umgebung. Die oberste Schicht $\mathbf{L}^t$ ist gefüllt mit Flüssigkeit. Die untere Ebene $\mathbf{L}^b$ enthält Sedimente (und Flüssigkeit). Die layer-Grenzen sind definiert durch die Funktionen η^t, η^m, η^b.

Relations between the horizontal flow velocity $\mathbf{U}^t$, the suspension concentration C^t and their average values $\mathbf{u}^t$, c^t in layer $\mathbf{L}^t$ are chosen in the form

(2) $\mathbf{U}^t(\mathbf{x},z,t)=h^t(\mathbf{x},t)\mathbf{u}^t(\mathbf{x},t)\psi(\mathbf{x},z,t),$

$$C^t(\mathbf{x},z,t)=h^t(\mathbf{x},t)c^t(\mathbf{x},t)\phi(\mathbf{x},z,t).$$

Here, the functions $\psi(\mathbf{x},z,t)$ and $\phi(\mathbf{x},z,t)$ define the vertical profile of the velocity and the suspension concentration.

The depth-averaged values are determined in the usual manner

(3) $\mathbf{u}^t(\mathbf{x},t)=\mathbf{P}\mathbf{U}^t,\ c^t(\mathbf{x},t)=\mathbf{P}C^t,$

where $\mathbf{P}$ is the operator of averaging

(4) $(\mathbf{P}F)(\mathbf{x},t)=\dfrac{1}{h^t}\displaystyle\int_{\eta^m}^{\eta^t}F(\mathbf{x},z,t)dz,\ h^t=\eta^t-\eta^m.$

It is obvious that the functions $\psi(\mathbf{x},z,t)$, $\phi(\mathbf{x},z,t)$ in (2) must satisfy to the conditions of normalization

(5) $h^t\mathbf{P}\psi=1,\ h^t\mathbf{P}\phi=1.$

If instead of the normalized profile $\psi(\mathbf{x},z,t)$, $\phi(\mathbf{x},z,t)$ the non-normalized profiles $\psi_0(\mathbf{x},z,t)$, $\phi_0(\mathbf{x},z,t)$ are known, then

(6) $\psi(\mathbf{x},z,t)=\dfrac{\psi_0(\mathbf{x},z,t)}{h^t\mathbf{P}\psi_0},\ \phi(\mathbf{x},z,t)=\dfrac{\phi_0(\mathbf{x},z,t)}{h^t\mathbf{P}\phi_0}.$

In principle, the function $\psi_0(\mathbf{x},z,t)$, $\phi_0(\mathbf{x},z,t)$ can be chosen arbitrary. However, it is preferable that these functions correspond to any solutions of the original equations.

Omitting the tedious procedure of averaging, we present the variant of the averaged equations, *i.e.*, the Saint-Venant-Exner equations (see, for example, ZHUKOV & SHIRYAEVA 2016):

continuity equation for the layer $\mathbf{L}^t$

(7) $\partial_t h^t+\mathrm{div}_{\mathbf{x}}(h^t\mathbf{u}^t)=0,$

equations of fluid motion for the layer $\mathbf{L}^t$

(8) $\partial_t(h^t\mathbf{u}^t)+\mathrm{div}_{\mathbf{x}}(\gamma_{\mathbf{u}}h^t(\mathbf{u}^t\otimes\mathbf{u}^t))=-gh^t\nabla_{\mathbf{x}}\eta^t-\boldsymbol{\sigma}_{\mathbf{t}}$

transport equation for suspension concentration in the layer of $\mathbf{L}^t$

(9) $\rho(\partial_t(h^tc^t)+\mathrm{div}_{\mathbf{x}}(\gamma_c h^tc^t\mathbf{u}^t))=Q_m,$

Exner equation

(10) $\rho_s(1-\varepsilon_p)\left(\partial_t h^b+\mathrm{div}_{\mathbf{x}}(h^b\mathbf{u}^b)\right)=-Q_m.$

Here, $\mathbf{u}^t(\mathbf{x},t)$ is the average fluid velocity in the horizontal direction at the layer $\mathbf{L}^t$, $\mathbf{u}^b(\mathbf{x},t)$ is the average fluid velocity in the horizontal direction at the layer $\mathbf{L}^b$, $c^t(\mathbf{x},t)$ is the averaged suspension concentration, ρ_s is the sediment density (not density of the continious media in the layer $\mathbf{L}^b$), $\eta^t(\mathbf{x},t)$ is the free boundary, $\eta^m(\mathbf{x},t)$ is the boundary between the layers, $\eta^b(\mathbf{x},t)$ is the bottom boundary, $\boldsymbol{\sigma}_{\mathbf{t}}$ is the shear stress at the boundary η^m between the layers, Q_m is the suspension flux at the boundary η^b between the layers, $h^t(\mathbf{x},t)$ is the top layer thickness, $h^b(\mathbf{x},t)$ is the bottom layer thickness, g is the gravity, ε_p is the porosity, $\mathrm{div}_{\mathbf{x}}$, $\nabla_{\mathbf{x}}$ are the 'plane' divergence and gradient.

Note that we use the Exner equation (10). in the form proposed in PAOLA & VOLLER (2005), see also FERNANDEZ-NIETO et al. (2014, 2015).

The coefficients γ_c and $\gamma_{\mathbf{u}}$ that arise in averaging have the form

(11) $\gamma_{\mathbf{u}}=h^t\displaystyle\int_{\eta^m}^{\eta^t}\psi^2(z)dz,\ \gamma_c=h^t\int_{\eta^m}^{\eta^t}\psi(z)\phi(z)dz.$

Note that γ_c, $\gamma_{\mathbf{u}}$ are not constant. These coefficients are depended on η^t, η^m, and the parameters that can contain the functions ψ, ϕ.

In equations (7) to (10) the thicknesses $h^t(\mathbf{x},t)$, $h^b(\mathbf{x},t)$, the suspension concentration $c^t(\mathbf{x},t)$, and the velocity $\mathbf{u}^t(\mathbf{x},t)$ are chosen as unknown quantities. We also assume that the parameters g, ρ_s, ε_p, and functions ψ, ϕ are assigned. The equations (7) – (10) are not closed, since it requires additional relations for shear stress $\boldsymbol{\sigma}_{\mathbf{t}}$, the flux Q_m on the boundary between the layers $\mathbf{L}^b$, $\mathbf{L}^t$, and the fluid velocity $\mathbf{u}^b$.

The averaged equations are obtained under the assumption of materiality of borders η^b, η^m, η^t. In other words, we consider that the boundaries are moving together with the continuous medium. The Exner equation (10) is presented for the case when the sediment density is assumed quasistationary and homogeneous in the vertical direction. It means that sediment density does not depend on z and coincides with the average sediment density ρ^b. In addition, we consider that the layer $\mathbf{L}^b$ is filled by heterogeneous continuous medium. In this case the average sediment density ρ^b can be written in the form $\rho^b = \rho_s(1-\varepsilon_p)$, where ε_p is porosity. More general variants of the Exner equation see, for example, in PAOLA & VOLLER (2005) and ZHUKOV & SHIRYAEVA (2016).

CONSTITUTIVE RELATIONS

The values $\boldsymbol{\sigma}_\mathbf{t}$, Q_m, $\mathbf{u}^b$ can be defined on the basis of the phenomenological (empirical) relations as done in most works (in particular, RIJN 1987, 1993, AUDUSSE et al. 2010, GAREGNANI 2011, FERNANDEZ-NIETO et al. 2014, 2015, MALDONADO & BORTHWICK 2016). In fact, the equations (7) to (10), and the coefficients $\gamma_\mathbf{u}$, γ_c are also possible to obtain on the basis of the phenomenological representations. However, from the mathematical point of view, it is more appropriate for the definition of $\boldsymbol{\sigma}_\mathbf{t}$, Q_m, $\mathbf{u}^b$ to use, where it is possible, only the physical-mechanical relations that in the procedure of averaging are arisen naturally from the original equations.

The equations (7) to (9) are obtained on the basis of the averaging method for hydrostatic approximation equations

(12) $\mathrm{div}_\mathbf{x}\mathbf{U}^t + \partial_z W^t = 0$, $\partial_z P^t = -\rho g$,

(13) $\rho\left(\partial_t \mathbf{U}^t + \mathbf{U}^t \cdot \nabla_\mathbf{x}\mathbf{U}^t + W^t\partial_z\mathbf{U}^t\right) = -\nabla_\mathbf{x}P^t + \rho\partial_z\left(K_m\partial_z\mathbf{U}^\mathbf{t}\right)$,

(14) $\rho(\partial_t C^t + \mathbf{U}^t \cdot \nabla_\mathbf{x} C^t + W^t\partial_z C^t) = -\partial_z Q^t$, $Q^t = -\rho(K_c\partial_z C^t + w_s C^t)$,

with boundary conditions

(15) $K_m\partial_z\mathbf{U}^t\big|_{z=\eta^t} = 0$, $\mathbf{U}^t\big|_{z=\eta^b+z_0} = 0$, $P^t\big|_{z=\eta^t} = \mathrm{const}$,

(16) $Q^t\big|_{z=\eta^t} = 0$, $Q^t\big|_{z=\eta^m} = Q_m \equiv -w_s(\rho C^t - \rho_s(1-\varepsilon_p))\big|_{z=\eta^m}$,

where $\mathbf{V}^\mathbf{t} = (\mathbf{U}^t, W^t)$ is the fluid velocity for original flow, $\mathbf{U}^\mathbf{t}$ is the horizontal velocity, W^t is the vertical velocity, P^t is the pressure, ρ is the density of the continuous media in the layer $\mathbf{L}^t$ (fluid density + suspension density), C^t is the suspension concentration, Q^t is the local suspension flux in the vertical direction, w_s is the settling velocity for particle of the suspension, z_0 is the roughness parameter, $K_m(z)$, $K_c(z)$ are the coefficients of turbulent viscosity and diffusion (e.g., MONIN & YAGLOM 1965, RIJN 1993, SHLIHTING 2006).

The first boundary condition (16) corresponds to absence of the suspension flux on the free boundary. The second bondary condition is proposed, in particular, in CHENG (1984), where it is called by the condition of non-equilibrium diffusive; see also RIJN (1987), where other boundary conditions are discussed. In fact, this condition is one of the constitutive relations and it identifies the concentration flux Q_m at the boundary η^m for equations (9), (10).

The shear stress $\boldsymbol{\sigma}_\mathbf{t}$ on the boundary between the layers after averaging of the term $\partial_z\left(K_m\partial_z\mathbf{U}^\mathbf{t}\right)$ (see (13)) is written as

(17) $\boldsymbol{\sigma}_\mathbf{t} = (K_m h^t \mathbf{u}^t \partial_z \psi(z))\big|_{z=\eta^m}$.

For further identification of $\boldsymbol{\sigma}_\mathbf{t}$, and coefficients $\gamma_\mathbf{u}$, γ_c the information about the functions ψ, ϕ is required. These functions can be obtained after identification of the coefficients of turbulent viscosity and diffusion from the kinematic approximation to the original equations in hydrostatic approximation.

We choose the coefficients of K_m, K_c in the form (here, we have used the Reynolds analogy (MONIN & YAGLOM 1965, SCHLICHTING 2006)

(18) $K_m = K_c = \kappa|\mathbf{U}_*|\dfrac{(z-\eta^t)(\eta^b - z)}{h^b + h^t}$, $\eta^m \le z \le \eta^t$,

where $\mathbf{U}_*$ is the effective average velocity, κ is the Kármán constant ($\kappa = 0.4$).

The justification for the selection of the relations (18) see, for example, in MONIN & YAGLOM (1965), GRISHANIN (1979) and SCHLICHTING (2006). Other forms of dependencies for the coefficients of turbulent viscosity and diffusion see, for example, in RIJN (1987, 1993).

Omitting inertial terms in the equations (12) to (14) (so called kinematic approximation) and solving these equations taking into account the boundary conditions (15), (16), we get that the horizontal velocity $\mathbf{U}^t$ has a logarithmic profile and the suspension concentration $C^t(z)$ has the Rouse profile

(19) $\mathbf{U}^t(\mathbf{x},z,t) = \dfrac{\mathbf{U}_*}{\kappa}\psi_0(\mathbf{x},z,t)$, $\eta^m \le z \le \eta^t$,

$C^t(\mathbf{x},z,t) = C^t(\mathbf{x},\eta^m,t)\left(\dfrac{h^b}{h^t}\right)^Z \phi_0(\mathbf{x},z,t)$, $\eta^b < \eta^m \le z \le \eta^t$,

where

(20) $\psi_0(\mathbf{x},z,t) = \ln\dfrac{z-\eta^b}{z_0}$, $\phi_0(\mathbf{x},z,t) = \left(\dfrac{\eta^t - z}{z-\eta^b}\right)^Z$.

Here, Z is the suspension number: $Z = w_s/(\kappa|\mathbf{U}_*|)$. Different formulas for settling velocity w_s are presented, for example, in DIETRICH (1982), RIJN (1993), CHENG (1997), BALDOCK et al. (2004), SONG ZHIYAO et al. (2008), and SOULSBY et al. (2013).

Substitution of the relations (18) into the formula (17) with the help (4), (6), (20) allows us to obtain the shear stress $\boldsymbol{\sigma}_{\mathbf{t}}$ in the form

(21) $$\boldsymbol{\sigma}_{\mathbf{t}} = (K_m h^t \mathbf{u}^t \partial_z \psi(z))\big|_{z=\eta^m} = \kappa^2 \Gamma_{\mathbf{t}} \,|\mathbf{u}^t|\, \mathbf{u}^t,$$

where $\Gamma_{\mathbf{t}}$ is the friction coefficient (see (24)).
Introducing parameters

(22) $$\beta = \frac{h^b}{z_0},\quad \mu = \frac{h^b + h^t}{h^b},\quad 1 < \mu < \infty,\quad \beta\mu > 1,$$

we present the relations for effective average velocity $|\mathbf{U}_*|$ and the coefficients $\Gamma_{\mathbf{t}}(\mu,\beta)$, $\gamma_{\mathbf{u}}(\mu,\beta)$, and $\gamma_c(\mu,\beta,Z)$

(23) $$|\mathbf{U}_*(\mathbf{x},t)| = \kappa\gamma_{\mathbf{e}}(\mu,\beta)\,|\mathbf{u}^t|,$$

$$\gamma_{\mathbf{e}} = \frac{(\mu-1)}{1-\mu-\ln\beta+\mu\ln\beta+\mu\ln\mu} > 0,$$

(24) $$\Gamma_{\mathbf{t}} = \frac{(\mu-1)^3}{\mu(1-\mu-\ln\beta+\mu\ln\beta+\mu\ln\mu)^2} > 0,$$

(25) $$\gamma_c = \frac{\mu-1}{1-\ln\beta-\mu+\mu\ln(\mu\beta)}\left(\ln\beta + \frac{I_2(\mu)}{I_3(\mu)}\right),$$

$$\gamma_{\mathbf{u}} = \frac{(\mu-1)((\mu-1)(\ln\beta-1)^2 + 2\mu\ln\beta\ln\mu + \mu(\ln\mu-1)^2 - 1)}{(\mu\ln\mu+\mu\ln\beta-\ln\beta-\mu+1)^2},$$

where

(26) $$I_2(\mu) = \int_1^{\mu}\left(\frac{\mu-\theta}{\theta}\right)^Z \ln\theta\, d\theta,\quad I_3 = \int_1^{\mu}\left(\frac{\mu-\theta}{\theta}\right)^Z d\theta.$$

The integrals $I_2(\mu)$, $I_3(\mu)$ are known as the Einstein integrals (EINSTEIN 1950), and can be calculated using hypergeometric functions, or infinite series (e.g., GUO & JULIEN 2004).

Finally, the velocity of the sediment in the layer of $\mathbf{L}^b$ can be defined by considering the equality of the tangential stresses at the boundary layers of $\mathbf{L}^b$, $\mathbf{L}^t$ (other relations see, for example, in RIJN 1987, 1993)

(27) $$\mathbf{u}^b = \frac{\kappa^2 \Gamma_{\mathbf{t}} h^b}{\nu_s}\,|\mathbf{u}^t|\,\mathbf{u}^t,$$

where ν_s is given viscosity of the fluid in the layer $\mathbf{L}^b$.

COMMENTS ON THE CHOICE OF THE BOUNDARIES η^b, η^m, η^t

Strictly speaking, equations (7) to (10) are not a closed system despite the fact that the constitutive relations are defined, for example, in the form (16), (21), (27). The fact that the functions γ_c, $\gamma_{\mathbf{u}}$, $\Gamma_{\mathbf{t}}$ depend on η^t, η^m, η^b. We note that η^t, η^m, η^b can not be determined uniquely by the thickness of layers as $h^t = \eta^t - \eta^m$, $h^b = \eta^m - \eta^b$ (see (1)). In addition, the equation (8) contains the term $gh^t\nabla_{\mathbf{x}}\eta^t$ which also includes the unknown function η^t. Thus, for the closing of the equations and unique definition of η^t, η^m, η^b at least one more equation or relation is required.

The simplest way to remove the indefiniteness of the location for the layers is the explicit definition of one of the functions $\eta^b(\mathbf{x},t)$, $\eta^m(\mathbf{x},t)$, $\eta^t(\mathbf{x},t)$. Note that in hydrology "adaptive" hypothesis of the sediment layer is sometimes used: the thickness of the sediment layer is assumed by proportional to the thickness of the liquid layer, see for example GRISHANIN (1979) and SEIN (1992). This hypothesis is unacceptable since the Exner equation (10) contradicts to the continuity equation (7) in the case when h^b is proportional to h^t.

Discussions of different variants of the boundary selection as a known function.

1) The boundary η^m between the layers is known.

If the boundary $\eta^m(\mathbf{x},t)$ is considered as a known function, then the equation (8) should be transformed to the form

(28) $$gh^t\nabla_{\mathbf{x}}\eta^t = gh^t\nabla_{\mathbf{x}}(h^t+\eta^m) = gh^t\nabla_{\mathbf{x}}h^t + gh^t\nabla_{\mathbf{x}}\eta^m.$$

In this case, the equation of motion (8) is exactly the same as the classic version of the shallow water equation.

The choice of the boundaries between the layers η^m as a known function is unsuccessful, since the position of this boundary is difficult to determine in practice. Strictly speaking, most models are used to determine this boundary, and the sedimentation problem is losing sense.

2) The bottom boundary η^b is known.

If the boundary $\eta^b(\mathbf{x},t)$ is considered as a known function, then the equation (8) should be transformed to the form

(29) $$gh^t\nabla_{\mathbf{x}}\eta^t = gh^t\nabla_{\mathbf{x}}(h^b+h^t) + gh^t\nabla_{\mathbf{x}}\eta^b.$$

In fact, the relation (29) dictates the choice of the new unknown function (h^b+h^t) instead h^b. In this case, to obtain an equation for a new variable (h^b+h^t) we need to add the equations (7) and (10). Otherwise, the system (7) to (10) is not a hyperbolic system, since the right hand side of (8) will contain derivatives of the unknown functions.

Selection of the boundary η^b, i.e., the elevation of the reservoir bottom as a known function, is quite reasonable and most often used in hydrology. However, if the system of equations (7) to (10) are used to study of the bottom morphology, the choice of η^b as unknown function is also not acceptable.

3) The free boundary η^t is known.

If the boundary $\eta^t(\mathbf{x},t)$ is considered as a known function, then the equation (8) does not need any transform. As a rule, the position of the free boundary η^t is also object of investigation. Selecting it as a known boundary is not acceptable. However, from the practical point of view, the position of the boundary η^t can be easily measured, for example, using tracking buoys.

KINEMATIC WAVES APPROXIMATION

To demonstrate the importance of the constitutive relation choice (21) we consider the spatially one-dimensional model based on the so-called kinematic wave approximation; see for example WHITHAM (1974). The main idea of this approximation is to neglect the inertial terms in the equations of fluid motion (8). From a physical point of view, this approximation corresponds to the steady-state flows, slowly changing over time. Note that this approximation may be not suitable in the case of turbid flows; see for example PARKER et al. (1986) and FALCINI et al. (2009). This allows us to determine the flow velocity by using algebraic relations. Spatial one-dimensional model is obtained from (7) to (27) by natural substitutions $\mathbf{u}^t(\mathbf{x},t)\to u^t(x,t)$, $\mathrm{div}_{\mathbf{x}}\to\partial_x$, and $\nabla_{\mathbf{x}}\to\partial_x$. We assume that free surface is known, *i.e.* the function $\eta^t(x,t)$ is given. As already mentioned, the choice of known η^t is caused because this boundary is the "observable", i.e. the displacement of the surface, in principle, can be measured.

Neglecting inertial terms in equation (8) and taking into account (21) we get a kinematic waves approximation

(30) $gh^t\partial_x\eta^t+\kappa^2\Gamma_{\mathbf{t}}\,|u^t\,|u^t=0$.

In this case the velocity u^t is obtained as

(31) $u^t=-\,\mathrm{sgn}(\partial_x\eta^t)\left(\dfrac{gh^t\,|\partial_x\eta^t\,|}{\kappa^2\Gamma_{\mathbf{t}}}\right)^{1/2}$.

The friction coefficient $\Gamma_{\mathbf{t}}$ depends on h^b, h^t and is defined by formulas (22), (24).

The kinematic waves approximation allows us to substantially simplify the equations (7) to (27) reducing them to equations for unknown function h^b, h^t, c^t only. On the basis of these equations, making different assumptions about the parameters, it is possible to construct a more simple asymptotic model of different levels of complexity. For example, a natural assumption that the thickness of sediment layer is much less than the thickness of the suspension layer

(32) $h^t>>h^b$

allows us to use for the coefficient $\Gamma_{\mathbf{t}}$ the following relation (see (24) as $\mu\to\infty$)

(33) $\Gamma_{\mathbf{t}}(h^b,h^t)\approx\Gamma_{\mathbf{t}}^a(h^t)=\left(\ln\dfrac{h^t}{z_0}\right)^{-2}$.

This is a very significant simplification of the model, since the velocity u^k becomes dependent only on h^t. In this case, the equation (7) is integrated independently from the equations (9), (10). We emphasize that the choice of the constitutive relations in the form (33) plays an important role for determining of the average velocity dependency on the problem parameters.

Next, we restrict the study of the behavior for only the thickness h^t. In other words we only investigate the equation which is obtained by substituting (31), (33) in equation (7). In this case, after substitutions

(34) $h^t=z_0h,\ t=\tau T_*,\ x=yL_*,$

$$p(y,\tau)=-\frac{T_*g^{1/2}}{L_*\kappa}\,\mathrm{sgn}(\partial_x\eta^t)\,|\,\partial_x\eta^t\,|^{1/2}\,(z_0)^{1/2},$$

we have equation

(35) $\partial_\tau h+\partial_y(p(y,\tau)h^{3/2}\ln h)=0$,

and initial condition

(36) $h(y,\tau)\,|_{\tau=0}=h_0(y)$.

Here, y, τ are the new coordinate and time, L_* is the scale length in horizontal direction, T_* is the scale time. The function $h_0(y)$ determines the initial thickness of the fluid layer. The function $p(y,\tau)$ is determined by the known (by assumption) free boundary $\eta^t(x,t)=\eta^t(yL_*,\tau T_*)$.

Problem (35), (36) can be easily solved by the method of characteristics; see WHITHAM (1974), which allows us to transform the Cauchy problem for partial differential equations of first order to the Cauchy problem for systems of ordinary differential equations.

We restrict our study to the problem when the free surface η^t has a constant slope

(37) $\partial_x\eta^t=\mathrm{const}<0$.

It is enough realistic situation. In the case when the thickness of the sediment layer is much less than the thickness of the suspension layer (see (32)) it is difficult to imagine that a thin sediment layer affects the free surface.

At appropriate choice of the scale time T_*, we get (see (34))

(38) $p(y,\tau)=1,\ T_*=\dfrac{L_*\kappa}{(gz_0\,|\,\partial_x\eta^t\,|)^{1/2}}$.

Then the solution of (35), (36) is reduced to the solution of the system

(39) $\dfrac{dH}{d\tau}=0,\ H(\tau)\,|_{\tau=0}=h_0(a),$

(40) $\dfrac{dY}{d\tau} = H^{1/2}\left(\dfrac{3}{2}\ln H + 1\right), \ Y(\tau)|_{\tau=0} = a.$

With the help of the functions $Y(\tau)$, $H(\tau)$ the problem solving (35), (36) can be written in implicit parametric form (e.g., WHITHAM 1974, ZHUKOV & SHIRYAEVA 2015)

(41) $h(y,\tau) = H(\tau; a) = h_0(a),$

$$y = Y(\tau; a) = \tau h_0^{1/2}(a)\left(\frac{3}{2}\ln h_0(a) + 1\right) + a.$$

Here, a is an arbitrary parameter.

The analysis of formula (41) shows that if for any values of parameter a the inequality $h_0'(a) < 0$ is valid, then at some time occurs 'rollover' of the function $h(y,\tau)$ profile and the shock waves are arisen (e.g., WHITHAM 1974).

In Figure 2 the results of the calculations based on the formula (41) and using the numerical solution of the problem (35), (36) with the help of the VoF method are presented.

For calculations we select the periodic perturbation of the layer thickness h_*

(42) $h_0(y) = h_* + b\sin(my), \ h_* = 3000, \ b = 300, \ m = 2.$

This corresponds to the case when the boundary between suspended impurity and sediment has two 'humps' and two 'cavities' with the amplitude b (0.5m) on the interval of length $2\pi L_*$ ($L_* = 100\text{m}$) and when the initial thickness of the liquid layer is h_* (5m). We note that corresponding dimension values of dimensionless quantities are indicated in brackets. Of course, this choice is made to simplify the calculations. Other parameters are selected to be suitable for straightforward riverbeds with a thin layer of silt.

(43) $|\partial_x \eta^t| = 0.096 \text{ m/km}, \ k_e = 0.050 \text{ m},$

(44) $z_0 = k_e/30 = 0.00167 \text{ m}, \ g = 9.8 \text{ m/s}^2, \ \kappa = 0.4.$

The friction coefficient is $\Gamma_t = 0.0156$ and the average velocity is $|\mathbf{u}^t| \approx 5 \text{ km/h}$ as $h = h_*$. The scale time is $T_* \approx 3.2 \cdot 10^4 \text{s}$.

Analysis of the results shows that from the periodic smooth of initial distribution (42) the shock waves are arisen at the short time (≈ 364 s). The shock waves velocity D is determined by the Rankine–Hugoniot condition, which in this case has the form

(45) $D = \dfrac{h_2^{3/2}\ln h_2 - h_1^{3/2}\ln h_1}{h_2 - h_1} \approx 712.265 \ (2.23\text{m/s}),$

where $h_2 \approx 3300$, $h_1 \approx 2700$ are the values of function across discontinuity.

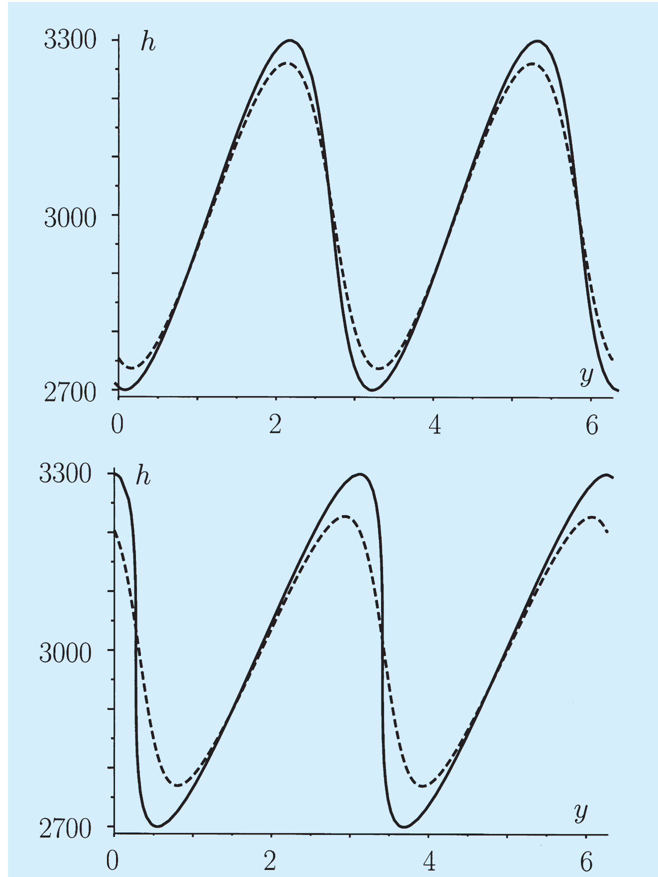

Fig. 2: The function $h(y,\tau)$ when $\tau = 0.0060, 0.0114$ (upper and lower pictures respectively). The dotted line corresponds to the solution that is obtained using the VoF method. In particular, this figure shows the profile overturn and the appearance of a shock waves.

Abb. 2: Die Funktion $h(y,\tau)$ bei $\tau = 0.0060, 0.0114$ (obere und untere Abbildung). Die gepunktete Linie entspricht der Lösung, mittels VoF-Methode erreicht wird. Insbesondere zeigt diese Abbildung wie das Profil kippt und die Erscheinung einer Stoßwelle annimmt.

CONCLUSION

The presented results show that VoF method allows to obtain a solution with a relative error of about 3 %. However the defects of this method are obvious. First, the waves amplitude is decreased over time, whereas for the exact solution the waves amplitude is unchanged. Secondly, the occurrence of discontinuous solutions is almost impossible to identify, since the solution is smoothed by the grid viscosity.

From the geomorphological point of view, the continuous profile overturn likely corresponds to the appearance of structures that mark an additional bottom shear stress due to shock waves (BRANDET et al. 1999, DROGHEI et al. 2016). Especially, we emphasize that this result can be obtained even for such a simple model, which is presented in this paper. This interesting phenomenon requires extensive additional investigation and will be the subject of next paper.

Finally we pay attention to the fact that for completeness of the model we used a variant when the behavior of the free boundary is known. Just a specific motion of the free boundary induces a spatial structure on the other boundaries. A selection of other boundaries as known functions can significantly change the results and requires additional study.

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