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On the Tangential Stresses at the Boundary Between the Layers for Two-Layer Sedimentation Models

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Abstract: It is shown that the exact value of the Chézy coefficient can be calculated on the basis of the two-layer model of fluid flow. This coefficient is determined by tangent stresses on the interface between the layers, which in turn are completely determined by the kinematic turbulent viscosity coefficient.

Zusammenfassung: In der Arbeit wird gezeigt, dass der genaue Wert der Chézy-Koeffizient berechnet werden kann auf der Grundlage des zwei-Schicht-Modells der Strömung. Dieses Verhältnis bestimmt die Tangenten der Spannung an der Grenze zwischen den Schichten. Wiederum richten sich die tangentiale Belastung nach den Koeffizienten der turbulenten kinematischen Viskosität.

INTRODUCTION

In hydrology it is common to study the sedimentation processes using a large number of semi-empirical correlations for the determination of sedimentation velocity, density of sediment particles, flux of sediment, and other parameters. These relations are usually obtained on the basis of experiments and some additional considerations, and are very different for fluvial flows, currents in seas and oceans, coastal flows, and so on.

It is difficult to list all the literature that discusses semi-empirical correlations. We mention only some monographs, reviews, and papers: EINSTEIN 1950, GRISHANIN 1979, van RIJN 1993, AMOUDRY & SOUZA 2001, BARYSHNIKOV 2007, AMOUDRY 2008, NADOLIN 2009, ZHUKOV & SHIRYAEVA 2015, 2016, NADOLIN & ZHILYAEV 2017.

In addition, almost all the works devoted to the construction of equations describing sedimentation processes contain certain defined semi-empirical relations as for example, AUDUSSE et al. 2010, FERNANDEZ-NIETO et al. 2014, 2015, GAREGNANI 2011, MALDONADO & BORTHWICK 2016.

The main purpose of this paper is the determination of the connection between semi-empirical correlations usually used in hydrology and the natural physicomechanical parameters for different mathematical models of the sedimentation. In particular in the study of sedimentation in fluid flows two-layer model is often used. The top layer is assumed filled by the homogeneous continuous medium consisting of the fluid and suspended sediment. The bottom layer is filled by a solid nonhomogeneous medium consisting of fluid and sediments.

In developing mathematical models, there is no need to use semiempirical correlations. In particular, it concerns the Chézy coefficient which is used in the formula to determine the average flow velocity in steady uniform turbulent open channel flow in the case of quadratic resistance.

In other words, it appears that many of the design formulas obtained on the basis of measurements and observations, in fact, can be derived with proper mathematical modeling from first principles. This allows you to construct a model for which all the physical laws are valid – in particular, conservation laws. In addition, when using such approaches, we can better understand the nature of physical processes.

ON THE CHÉZY COEFFICIENT

In hydrology the Chézy formula is often used for determining the average velocity $\bar{u}$ of the steady uniform liquid motion in the case of the quadratic resistance

(1) $\quad \bar{u}^2 = C^2 HI,$

where H is the depth of the fluid (m), I is the slope of the free surface ($\mathrm{m/m}$), C is the Chézy coefficient ($\mathrm{m}^{1/2}\cdot\mathrm{s}^{-1}$).

The Chézy coefficient is considered as an empirical quantity that characterizes the fluid flow for steady-state turbulent flow. In fact, in mathematical modeling of fluid flow, there is no need to consider the Chézy coefficient as an empirical quantity. Below on the example of turbulent flow is shown that the value of C is completely determined by the tangential stress on the boundary between the fluid flow and the "bottom" of the flow. The scheme illustrating the calculation of the Chézy coefficient is shown on Figure 1.

Here we present only the main result (see details in the Appendix). In the case of the domain shown on Figure 2 the kinematic waves approximation for the averaged equations is written in the form (see (32))

(2) $\quad gh^t\nabla_{\mathbf{x}}\eta^t + \boldsymbol{\sigma}_{\mathbf{t}} = 0,$

(3) $\quad \boldsymbol{\sigma}_{\mathbf{t}} = \kappa^2\Gamma_{\mathbf{t}}\,|\mathbf{u}|\,\mathbf{u},$

where g is the gravity ($\mathrm{m}\cdot\mathrm{s}^{-2}$), h^t is the thickness of fluid layer (m), $\mathbf{u}$ is the depth-averaged velocity of fluid ($\mathrm{m}\cdot\mathrm{s}^{-1}$), κ is the Kármán's constant, η^t is the function specifying the free surface $z=\eta^t(\mathbf{x},t)$, $\nabla_{\mathbf{x}}=(\partial_x,\partial_y,0)$, $\boldsymbol{\sigma}_{\mathbf{t}}$ is the tangent stress at the boundary $z=\eta^m(\mathbf{x},t)$ ($\mathrm{m}^2\cdot\mathrm{s}^{-2}$), $\Gamma_{\mathbf{t}}$ is the friction coefficient.

Keywords: depth averaged model, friction coefficient, kinematic waves approximation.

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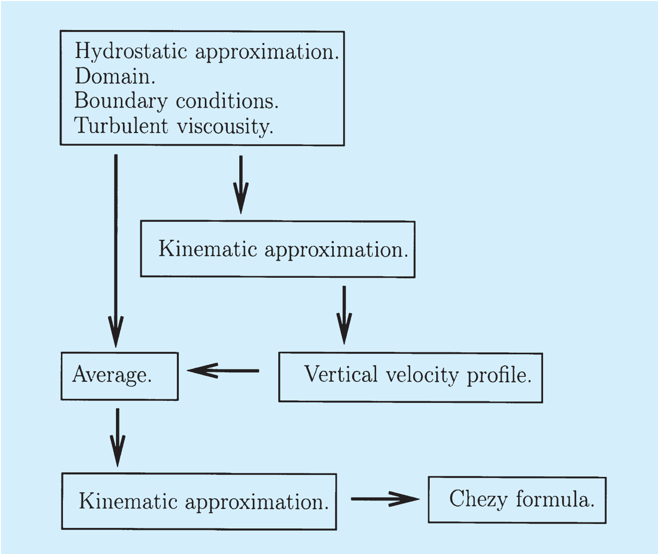

Fig. 1: The scheme for constructing of the Chézy formula. Sequence of actions, which are used for derivation of the Chézy formula.

Abb. 1: Schema der Konstruktion der Chézy-Formel. Ablauf von Aktionen für die Ausgabe der Chézy-Formel.

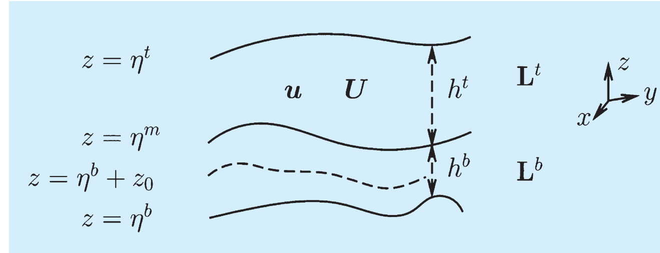

Fig. 2: The layers of the continuous medium. The top layer $\mathbf{L}^t$ is filled by fluid. The bottom layer $\mathbf{L}^b$ contains sediments (and fluid). The layer boundaries are defined by the functions η^t, η^m, η^b. Here, z_0 is the roughness parameter specifying the surface $z = \eta^b + z_0(\mathbf{x}, t)$ where velocity $\mathbf{U}(\mathbf{x}, t)$ becomes zero.

Abb. 2: Schichten der ununterbrochenen Umgebung. Die oberste Schicht $\mathbf{L}^t$ ist gefüllt mit Flüssigkeit. Die untere Ebene $\mathbf{L}^b$ enthält Sedimente (und Flüssigkeit). Die layer-Grenzen sind definiert durch die Funktionen η^t, η^m, η^b. Hier wird z_0 der Rauheitparameter, der die Oberfläche $z = \eta^b + z_0(\mathbf{x}, t)$ angibt wo Geschwindigkeit $\mathbf{U}(\mathbf{x}, t)$ null wird.

To obtain the required formulas, we use a two-layer model. We assume that the domain consists of two layers of $\mathbf{L}^t$, $\mathbf{L}^b$ (see Fig. 2). The top layer $\mathbf{L}^t$ is filled by fluid. The bottom layer $\mathbf{L}^b$ contains sediments (and fluid). The layer boundaries are defined by the functions η^t, η^m, η^b.

(4) $$\mathbf{L}^t = \{(\mathbf{x}, z) : \eta^m(\mathbf{x}, t) \le z \le \eta^t(\mathbf{x}, t)\},$$
$$\mathbf{L}^b = \{(\mathbf{x}, z) : \eta^b(\mathbf{x}, t) \le z \le \eta^m(\mathbf{x}, t)\},$$
$$h^t = \eta^t - \eta^m, \quad h^b = \eta^m - \eta^b, \quad \mathbf{x} = (x, y).$$

Here, $h^t(\mathbf{x}, t)$, $h^b(\mathbf{x}, t)$ are the layers thickness (Fig. 2).

For flow with constant slope I of the free surface $z = \eta^t$

(5) $$I = -\nabla_{\mathbf{x}} \eta^t,$$

using (2), (3) we obtain a formula coinciding with Chézy formula (1)

(6) $$|\mathbf{u}|^2 = C^2 h^t I,$$

where

(7) $$C = \frac{g^{1/2}}{\kappa} \frac{1}{\sqrt{\Gamma_{\mathbf{t}}}}.$$

The friction coefficient $\Gamma_{\mathbf{t}}$ has the form

(8) $$\Gamma_{\mathbf{t}}(\mathbf{x}, t) = \frac{(h^t)^3}{(h^b + h^t)\left(\int_{\eta^m}^{\eta^t} \psi_0(\mathbf{x}, z', t)\, dz'\right)^2},$$

where the function $\psi_0(\mathbf{x}, z, t)$ determines a vertical profile of horizontal (not average) velocity (see (22))

(9) $$\mathbf{U}(\mathbf{x}, t) = \frac{\mathbf{U}_*(\mathbf{x}, t)}{\kappa} \psi_0(\mathbf{x}, z, t), \quad \eta^m \le z \le \eta^t,$$

(10) $$\psi_0(\mathbf{x}, z, t) = \ln \frac{z - \eta^b}{z_0}.$$

Here, z_0 is the roughness parameter (m) specifying the surface $z = \eta^b + z_0(\mathbf{x}, t)$ (Fig. 2) where velocity $\mathbf{U}(\mathbf{x}, t)$ becomes zero, $\mathbf{U}_*(\mathbf{x}, t)$ is the characteristic of flow velocity.

In turn, formulas (9), (10) are valid in the case where the turbulent viscosity coefficient is specified as

(11) $$K_m = K_* \frac{(z - \eta^t)(\eta^b - z)}{h^b + h^t}, \quad \eta^m \le z \le \eta^t,$$

(12) $$K_* = \kappa U_*,$$

where U_* is the characteristic of turbulent viscosity.

We note that the formula (8) is written for a two-layer model, but it is also valid for single layer model. In this case, we assume that $z_0 = h^b$ and the surface $z = \eta^m$ specifies the level at which the flow velocity becomes equal to zero.

ASYMPTOTICS OF THE FRICTION COEFFICIENT

In the case of uniform steady-state turbulent flow there is a large number of approximate formulas for the Chézy coefficient. For example, in (RIJN 1993) it is given by the formula

(13) $$C = \frac{g^{1/2}}{\kappa}\left(\ln \frac{H}{z_0} - 1\right),$$

where $H = h^b + h^t$ is the total thickness of the liquid layers (see Fig. 2).

The indicated formula is valid for single-layer models, when the boundary between the layers $z = \eta^m$ is identified with the surface $z = \eta^b + z_0$, which specifies the roughness ($h^b = z_0$). Usually, instead of (13) the approximate formulas for specific values of the roughness parameter are used. For example, as $z_0 = k_e/30$, (k_e is the effective roughness (m)), $g = 9.81$, $\kappa = 0.4$, the following formula is used:

(14) $$C \approx 18 \lg \frac{12H}{k_e}.$$

Formulas for other values of z_0 are given in (RIJN 1993). It should be noted that in (RIJN 1993), as in other literary sources, it is unclear why, instead of the exact formula (13), the approximate values, similar to the formula (14), are used. In this case the rounding leads to a wrong result. In particular, if we use more accurate formula (13), then instead of formula (14) we get

$$C \approx 18 \lg \frac{11H}{k_e}.$$

Formula (14) is valid in the case when $z_0 \approx 0.0307 k_e$ is selected instead of $z_0 = k_e/30 \approx 0.0333 k_e$.

Using the formulas (7), (30), (31), it is easy to show that at $h^b = z_0$, $\beta = 1$ we have

$$(15) \quad C \approx \frac{g^{1/2}}{\kappa}\left(\ln \mu - 1 + \frac{1}{\mu}\right), \quad \frac{H}{z_0} = \frac{h^t + z_0}{z_0} = \mu \to \infty,$$

that coincides with (13).

APPENDIX

Following the scheme of constructing the Chézy formula (Fig.1), we present a detailed relation, which allow us to calculate Chézy coefficient (or the friction coefficient $\Gamma_{\mathbf{t}}$).

The hydrostatic approximation

The incompressibility equation

$$(16) \quad \mathrm{div}_{\mathbf{x}} \mathbf{U} + \partial_z W = 0, \quad \mathbf{V} = (\mathbf{U}, W), \quad \mathbf{U} = (U, V).$$

The equations of the fluid motion (in the horizontal direction)

$$(17) \quad \partial_t \mathbf{U} + \mathbf{U} \cdot \nabla_{\mathbf{x}} \mathbf{U} + W \partial_z \mathbf{U} = = -\frac{1}{\rho} \nabla_{\mathbf{x}} P + \partial_z (K_m \partial_z \mathbf{U}).$$

The equations of hydrostatics

$$(18) \quad \partial_z P = -\rho g.$$

Here, $\mathbf{V}$ is the velocity, $\mathbf{U}$ is the horizontal velocity, W is the vertical velocity, P is the pressure, K_m is the coefficient of turbulent viscosity, g is the gravity.

Domain

The domain is given by relations (4) (see also Fig. 2).

The boundary conditions

We choose the condition of absence of tangential stresses and constant pressure on the free boundary (so called dynamic conditions) as boundary conditions

$$(19) \quad K_m \partial_z \mathbf{U}\big|_{z=\eta^t} = 0, \quad P\big|_{z=\eta^t} = \text{const}.$$

In addition, we assume that for a surface $z = \eta^b + z_0$ velocity is zero

$$(20) \quad \mathbf{U}\big|_{z=\eta^b + z_0} = 0,$$

where z_0 is the roughness parameter (m).

The turbulent viscosity coefficient.

The turbulent viscosity coefficient K_m is given by formulas (11), (12). Recall that K_m may depend on time and coordinates.

The kinematic approximation.

The kinematic approximation of the equations (16), (17), (18) means neglecting of the inertial terms in the equations (17) (other equations stay unchanged)

$$(21) \quad -\frac{1}{\rho} \nabla_{\mathbf{x}} P + \partial_z (K_m \partial_z \mathbf{U}) = 0.$$

The vertical velocity profile

The solution of the problem (16), (21), (18), (19), (20), (11), (12) is well-known in the theory of turbulence (LANDAU & LIFSHITZ 1986, MONIN & YAGLOM 1965, SCHLICHTING 2006). Usually this solution is written in the following form (cf. (9), (10))

$$(22) \quad \mathbf{U} = \frac{\mathbf{U}_*}{\kappa} \psi_0(\mathbf{x}, z, t), \qquad \psi_0(\mathbf{x}, z, t) = \ln \frac{z - \eta^b}{z_0},$$

$\eta^m \le z \le \eta^t$, where $\mathbf{U}_*$ is characteristic velocity, κ is the Kármán's constant ($\kappa \approx 0.41$).

In principle, we can consider the roughness parameter z_0 as the function $z_0 = z_0(\mathbf{x}, t)$, which must satisfy the inequalities $\eta^b < \eta^b + z_0 < \eta^t$ or $0 < z_0 < h^t + h^b$.

The function $\psi_0(z)$ determines a vertical profile of flow. It is convenient to use a normalized function

$$\psi(\mathbf{x}, z, t) = \frac{\psi_0(\mathbf{x}, z, t)}{\int_{\eta^m}^{\eta^t} \psi_0(\mathbf{x}, z', t) dz'}, \qquad \int_{\eta^m(\mathbf{x},t)}^{\eta^t(\mathbf{x},t)} \psi(\mathbf{x}, z, t) dz = 1$$

which has the form (arguments $\mathbf{x}$, t are omitted for brevity)

$$(23) \quad \psi(z) = \frac{\ln \dfrac{z - \eta^b}{z_0}}{(h^b + h^t) \ln \dfrac{h^b + h^t}{z_0} - h^b \ln \dfrac{h^b}{z_0} - h^t}.$$

Average

Information about the function $\psi(z)$ allows us to perform the procedure of the averaging for original problem (16), (17), (18), (19), (20), (11), (12). We are looking for a solution in the form

$$(24) \quad \mathbf{U}(\mathbf{x}, z, t) = h^t \mathbf{u}(\mathbf{x}, t) \psi(z),$$

where $\mathbf{u}$ is the average horizontal velocity, which is given by

$$(25) \quad \mathbf{u}(\mathbf{x}, t) = \frac{1}{h^t} \int_{\eta^m}^{\eta^t} \mathbf{U}(\mathbf{x}, z, t) dz.$$

The system of averaged equations

To construct the averaged equations we integrate (16), (17), (18) over thickness of the layer $\mathbf{L}^t$ taking into account the conditions (19), (20), and the relations (24), (25). We also assume that all the boundaries η^b, η^m, η^t are material (moving together with the continuous medium). Omitting cumbersome transformations, we give the final result.

The incompressibility equation for a layer of $\mathbf{L}^t$

(26) $\partial_t h^t + \mathrm{div}_{\mathbf{x}}(h^t\mathbf{u}) = 0$.

The equation of the fluid motion for the layer $\mathbf{L}^t$

(27) $\partial_t(h^t\mathbf{u}) + \mathrm{div}_{\mathbf{x}}(\gamma_{\mathbf{u}} h^t(\mathbf{u}\otimes\mathbf{u})) = -gh^t\nabla_{\mathbf{x}}\eta^t - \boldsymbol{\sigma}_{\mathbf{t}}$,

(28) $\boldsymbol{\sigma}_{\mathbf{t}} = (K_m h^t\mathbf{u}\partial_z\psi(z))\big|_{z=\eta^m}$,

where the coefficient $\gamma_{\mathbf{u}}(\mathbf{x},t)$ is determined by the relation

(29) $\gamma_{\mathbf{u}} = h^t\int\limits_{\eta^m}^{\eta^t}\psi^2(\mathbf{x},z,t)dz$.

Tangent stress at the boundary

The value of $\boldsymbol{\sigma}_{\mathbf{t}}$ is the tangent stress at the boundary between the layers $\mathbf{L}^t$ and $\mathbf{L}^b$, whose dependence on the average velocity $\mathbf{u}$ is determined by the turbulent viscosity coefficient K_m and the vertical velocity profile $\psi(z)$. In fact, $\boldsymbol{\sigma}_{\mathbf{t}}$ is determined completely by the coefficient K_m, since $\psi(z)$ is also determined on the basis of information about the coefficient of turbulent viscosity. Substituting (11), (12) and (23) into (28) we get (3) formula

$\boldsymbol{\sigma}_{\mathbf{t}} = \kappa^2\Gamma_{\mathbf{t}}|\mathbf{u}|\mathbf{u}$,

and the relation (8) for the friction coefficient $\Gamma_{\mathbf{t}}$, which is convenient to write in the form

(30) $\Gamma_{\mathbf{t}} = \dfrac{(\mu-1)^3}{\mu(1-\mu-\ln\beta+\mu\ln\beta+\mu\ln\mu)^2}$,

(31) $\beta = \dfrac{h^b}{z_0}$, $\mu = \dfrac{h^b+h^t}{h^b}$, $1<\mu<\infty$, $0<\dfrac{1}{\beta}<\mu$.

The dependence of the normalized coefficient Chézy $C\kappa g^{-1/2} = 1/\sqrt{\Gamma_{\mathbf{t}}}$ on the parameters μ, β is shown in Figure 3.

Kinematic approximation of the averaged equations

As in the case of complete equations, the kinematic approximation means omitting the inertial terms in the equation fluid motion (27) (cf. (2))

(32) $gh^t\nabla_{\mathbf{x}}\eta^t + \boldsymbol{\sigma}_{\mathbf{t}} = 0$.

The method of the Chézy coefficient calculation with the help (32) is already described earlier (see (7)). We note that if we restrict the calculation of the Chézy coefficient, there is no need to use averaging of the original equations. To obtain (32), it is sufficient to apply the averaging only to kinematic approximation (21).

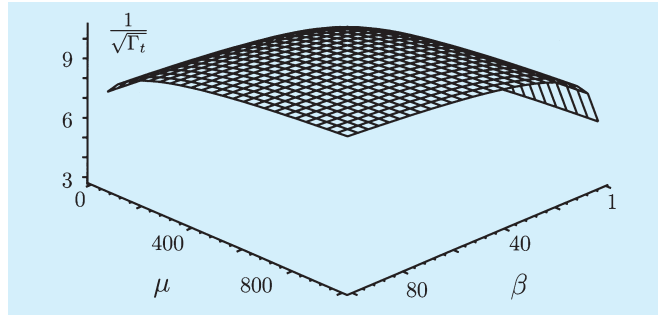

Fig. 3: The dependence of the Chézy coefficient $C\kappa g^{-1/2} = 1/\sqrt{\Gamma_{\mathbf{t}}}$ on the parameters μ, β.

Abb. 3: Abhängigkeit des Chézy Koeffizienten $C\kappa g^{-1/2} = 1/\sqrt{\Gamma_{\mathbf{t}}}$ von Parametern μ, β.

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